be claimed for the form in (31), and that the purpose of this section is to indicate the special nature of the difficulties associated with the near-degenerate cases, and why the attempted expansion runs into trouble.

X. Conclusions

With the publication of this paper (and also a forthcoming paper dealing with propagation of the TM modes [3]) we can say that the problem of propagation in a twisted rectangular guide is complete and well understood with the exception that there is still work that needs to be done to obtain a valid and useful expansion in the neighborhood of, but not exactly at, a degeneracy.

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Dispersion Relations for Comb-Type Slow-Wave Structures

I. L. VERBITSKII

Abstract—**Asymptotically accurate dispersion relations for slow eigenwaves of a dense comb placed on the wall of a parallel-plate waveguide are given in closed form. The equations can be easily resolved numerically.**

An analysis of dispersion relations for combs, based on the above-mentioned equations, has advantages over commonly used methods because of the simplicity of the necessary calculations and clarity of results.

I. Introduction

Dispersion relations for the widely used comb-type slow-wave structures usually are obtained through rather complicated computations [1], [2], [3]. In this communication asymptotically accurate dispersion relations for slow eigenwaves of a dense comb placed on the wall of a parallel-plate waveguide are given in closed form. The equations can be easily resolved numerically. If the ratio of the light velocity c to that of the slow wave v is not too small (e.g., $c/v \geqslant 2.5$), explicit formulas for the wavelength as a function of the phase shift can be obtained.

An analysis of dispersion relations for combs, based on the above-mentioned equations, has advantages over commonly used methods because of the simplicity of the necessary calculations and the clarity of results.

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II. Basic Relations

Consider a comb placed on the wall of a parallel-plate waveguide (Fig. 1), where d is the period, h the groove depth, δ the groove width, and A the spacing between tops of the lamellas and the upper waveguide wall. We will also use the following notation: λ for the free-space wavelength, ω the circular frequency, $k=2\pi/\lambda=\omega/c$ the wavenumber, $\kappa = d/\lambda = kd/2\pi$, $\theta = \delta/d$, $\nu = \kappa\theta \tan kh$, β the phase constant of the slow wave, $\alpha=\sqrt{\beta^2-k^2}$ the transverse wavenumber of the slow wave, $b=\beta d/2\pi$ (where βd is the phase shift over one period), and $a=\alpha d/2\pi$.

For the case of $A=\infty$, open comb, the dispersion equation for the TM slow wave (with nonzero components E_x, E_y, and H_z) was obtained in [4, eq. (16)] with an assumption that terms of the order $\exp(-2\pi h/\delta)$ and κ^2 could be neglected (indeed, these values are generally very small in real slow-wave structures). Through some laborious and sophisticated calculations the author has succeeded in obtaining an explicit expression for the integral [4, eq. 1(b)] from (the derivation has been omitted here). With the aid of this formula and an additional assumption, i.e., $\exp(-2\pi A/d)\ll 1$ (which holds in most cases), we can now obtain a closed form of the dispersion equation for slow waves in a comb placed in a waveguide.

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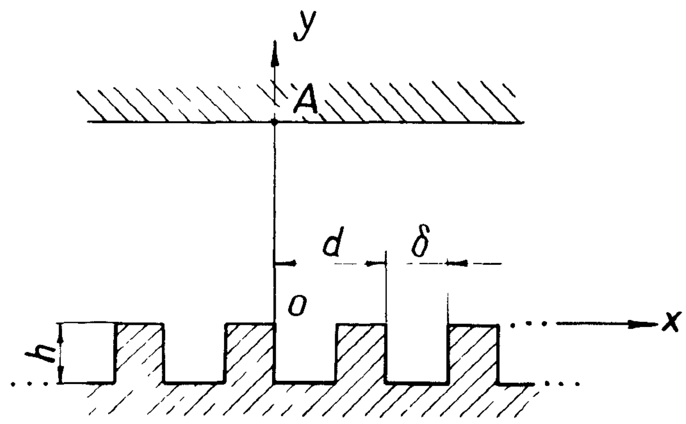

Fig. 1.

This relation is

$$\frac{1}{\nu}=\frac{1}{b}-\varphi(b)-f(\theta)-\frac{2}{b}\frac{[f-a+(a+b)\exp(-2\alpha A)]\mathcal{P}}{[b-a-(a+b)\exp(-2\alpha A)]\mathcal{P}-a-b-(b-a)\exp(-2\alpha A)} \tag{1}$$

where

$$\mathcal{P}=\left[(1-\theta)^{1-\theta}(1+\theta)^{1+\theta}\right]^{-2b}\det\exp(-2b\mu(\theta)),$$

$$f(\theta)=2\ln(4\theta)+\frac{1-\theta}{\theta}\ln(1-\theta)-\frac{1+\theta}{\theta}\ln(1+\theta),$$

$$\varphi(b)=\psi(1+b)+\psi(1-b)+2C,$$

$$\psi(z)=\Gamma'(z)/\Gamma(z)$$

is the digamma function and $C=0.5772\cdots$ is Euler's constant. Equation (1) has been derived by taking into account the reflection from the upper wall of the waveguide only for the fundamental harmonic of the surface field. Reflections of higher harmonics are negligible because of the imposed condition $\exp(-2\pi A/d)\ll 1$ (see [5, ch. 51, p. 266].

Equation (1) can be easily resolved numerically. Detailed tables [6] and effective methods of calculation are known for the one special function involved $\psi(z)$. Using the familiar expansions of $\psi(z)$ one can obtain the following approximate expression for $\varphi(b)$:

$$\varphi(b)\simeq 3-\frac{2}{2-b^2}-\frac{4}{4-b^2}-0.158b^2. \tag{2}$$

The accuracy is of the order 10^{-4} with $|b|\leqslant 0.5$.

III. Particular Cases

For particular cases (1) can be simplified. Thus, for surface waves in an open comb ($A=\infty$, $\exp(-2\alpha A)=0$) equation (1) reduces to

$$\frac{1}{\nu}=\frac{1}{b}-\varphi(b)f(\theta)+\frac{2(b-a)\mathcal{P}}{b[(a-b)\mathcal{P}+a+b]} \tag{3}$$

with $a\gg\kappa^2$ (i.e., the ratio c/v is not too small) the following equation can be deduced from (3):

$$\frac{1}{\nu}=\frac{1}{a}-\varphi(b)-f(\theta). \tag{4}$$

At $b\simeq a$ (i.e., c/v is large) equation (4) can be reduced to a still simpler form; viz.,

$$\frac{1}{\nu}=\frac{1}{b}-\varphi(b)-f(\theta). \tag{4a}$$

The general equation (1) with $b\approx a$ takes the form

$$\frac{1}{\nu}=\frac{1}{b}\coth b\left[\bar{A}+\mu(\theta)\right]-\varphi(b)-f(\theta) \tag{1a}$$

where $\bar{A}=2\pi A/d$. It can be seen from (1a) that $\mu(\theta)$ accounts for penetration of the quasi-static field into the grooves, which effect is the same as though the boundary $y=0$ were displaced to the position $y=-(d/2\pi)\mu(\theta)$. A similar effect was described in papers [7] and [8] for a purely static case.

With the same accuracy as used in deriving (1a) and (4a) we can write their approximate closed-form solutions, namely

$$\kappa=\frac{(n+0.5)\pi}{\bar{h}+\theta\left\{\frac{1}{b}\coth b\left[\bar{A}+\mu(\theta)\right]-\varphi(b)-f(\theta)\right\}},\qquad n=0,1,2,\cdots \tag{1b}$$

and

$$\kappa=\frac{(n+0.5)\pi}{\bar{h}+\theta\left[\frac{1}{b}-\varphi(b)-f(\theta)\right]},\qquad n=0,1,2,\cdots \tag{4b}$$

where $\bar{h}=2\pi h/d$. Different values of n here correspond to different passbands of the structure and (1b) and (4b) are valid only for those n's which provide for sufficiently large values of the ratio c/v (i.e., $a\approx b$, $\kappa^2\ll b^2$). Taking $n=0$ we obtain the necessary conditions of low-phase velocities as

$$\left\{\frac{d}{4bh+\delta\left[\coth b(\bar{A}+\mu)-b\varphi(b)-bf(\theta)\right]}\right\}^2\ll 1$$

for the comb in the waveguide and $(2\beta h/\pi)^2\ll 1$ for the open comb.

Substituting $b=1/2$ in (4b) we find cutoff wavelengths

$$\lambda=\frac{2h+dg(\theta)/\pi}{n+0.5}.$$

With $n=0$ the cutoff depth of the fundamental band becomes

$$h=\frac{\lambda}{4}-\frac{d}{2\pi}g(\theta)$$

where

$$g(\theta)=2\theta\ln\theta-(1-\theta)\ln(1-\theta)+(1+\theta)\ln(1+\theta).$$

For vanishingly thin lamellas the earlier known results follow immediately: $\kappa\tan\kappa h=1/2\ln 2$ [9], $h=\lambda/4-(d/\pi)\ln 2$ [10].

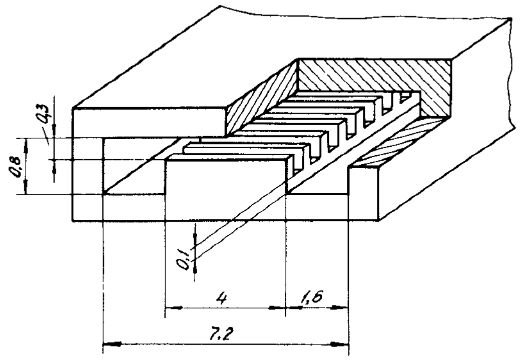

Fig. 2.

IV. Waves in the Stopband

Representing (4a) in the form

$$\frac{1}{\nu} = -\left[\psi(b)+\psi(1-b)+f(\theta)\right]$$

we can identify real solutions ν with $b=i/2+it$, such that $g(\theta)\geqslant 1/\nu > -\infty$ at $0\leqslant t \geqslant \infty$. These solutions correspond to surface waves in the stopband, traveling along the structure with a frequency-independent phase shift $\beta d=\pi$ and attenuation t. The corresponding wavelength varies between

$$\lambda = \frac{2h+dg(\theta)/\pi}{n+0.5}, \qquad \text{at } t=0$$

and

$$\lambda = \frac{2h}{n+1}, \qquad \text{at } t=\infty.$$

For $\theta = I$ such waves were described in [9].

In the vicinity of the point $\beta=\kappa$, $a=0$ (where the surface wave transforms to the bulk wave) the following relations can be obtained from (1):

$$\kappa \approx \frac{n\pi}{\bar{h}-\mu(\theta)/\theta}$$

or

$$h \approx n\frac{\lambda}{2}+\frac{d\mu(\theta)}{2\pi\theta}, \qquad n=1,2,\cdots.$$

V. Evaluation of Accuracy

To evaluate numerical accuracy of the above results, a comparison has been made with some known numerical analyses [1]–[3]. The error in solutions of (3) proved to be below 0.5 percent in all the cases considered. The error of the results obtained using (4) is less than 1 percent with $c/v \geqslant 2$; also, the error is less than 1 percent using the explicit formula (4b) as long as $c/v \geqslant 2.5$. For $c/v \geqslant 3$ the error of the latter expression is less than 0.5 percent. It might be noteworthy that the "two-dimensional" theory developed is valid for real three-dimensional structures in some domains of phase shifts. In the experimenting with a prototype BWO whose interaction space is shown on Fig. 2 (i.e. $d=0.19$ mm; $\delta=0.1$ mm; and $\theta=0.527$), oscillations were excited at $\beta d \geqslant \pi/4$. Fig. 3 presents a theoretical wavelength–voltage dependence of the wave excited (solid line) as given by (1) with the assumption that the wave phase velocity and electron beam velocity coincide. The circles represent experimental data. It should be observed that the measured and the calculated values are in good agreement.

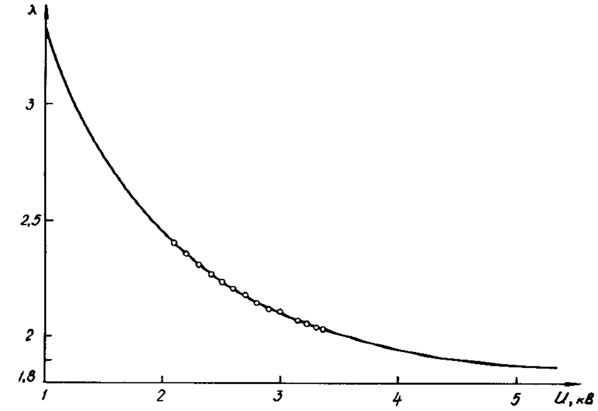

Fig. 3.

VI. Conclusion

A simple closed form has been obtained for the dispersion relations in comb-type slow-wave structures. The formulas are characterized by a high accuracy and include many earlier results as particular cases. The results obtained allow one to analyze the field near and beyond the cutoff.

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